CO-FROBENIUS HOPF ALGEBRAS AND THE CORADICAL FILTRATION

NICOLÁS ANDRUSKIEWITSCH AND SORIN DĂSCĂLESCU

ABSTRACT. We prove that a Hopf algebra with a finite coradical filtration is co-Frobenius, *i. e.* there is a non-zero integral on it. As an application, we show that algebras of functions on quantum groups at roots of one are co-Frobenius. We also characterize co-Frobenius Hopf algebras with coradical a Hopf subalgebra. This characterization is in the framework of the lifting method, due to H.-J. Schneider and the first author. Here is our main result. Let H be a Hopf algebra whose coradical is a Hopf algebra. Let gr H be the associated graded coalgebra and let R be the diagram of H, *c. f.* [3]. Then the following are equivalent: (1) H is co-Frobenius; (2) gr H is co-Frobenius; (3) R is finite dimensional; (4) the coradical filtration of H is finite. This Theorem allows to construct systematically examples of co-Frobenius Hopf algebras, and opens the way to the classification of ample classes of such Hopf algebras.

0. INTRODUCTION AND PRELIMINARIES

Among the many similarities between the theory of Hopf algebras and the theory of groups, the notion of "integral" occupies a central place. Here, recall that a "left integral" over a Hopf algebra H is a linear map $\int \in H^*$ which is left invariant; that is, $\alpha \int = \alpha(1) \int$ for all $\alpha \in H^*$. If H is the algebra of regular functions on a compact Lie group G, this is exactly (the restriction of) a left Haar measure on G. Several basic results on integrals are known, see [14, Ch. 4]. In particular, the space of left integrals has dimension ≤ 1. A fundamental problem is to determine the class of Hopf algebras having a non-zero integral. These are called *co-Frobenius* Hopf algebras, as explained below. Classically, finite dimensional Hopf algebras [18] and cosemisimple Hopf algebras [27] are co-Frobenius. The last class of examples contains the "compact quantum groups" defined by Woronowicz [28]; an alternative proof of the existence of a non-zero Haar measure on a compact quantum group is provided in *loc. cit.* It is also known that the algebra of regular functions on an algebraic group G, in characteristic 0, is co-Frobenius if and only if the group is reductive [25].

Examples of co-Frobenius Hopf algebras do not abound in the literature. Clearly, the tensor product of two co-Frobenius Hopf algebras also is; more generally, a cleft extension of two co-Frobenius Hopf algebras also is co-Frobenius [8]. Recently, several new examples of quantum groups with non-zero integrals have been discovered, including liftings of finite quantum linear spaces over an abelian group [7]. These are infinite dimensional pointed Hopf algebras which

are not cosemisimple. See some more examples in [17]. One of the corollaries of the main results in this paper is the systematic construction of many more new examples of co-Frobenius Hopf algebras.

The experience of the last years shows that an efficient approach to Hopf algebras with non-zero integrals is the one from a comodule theory point of view. For instance an easy conceptual proof for the uniqueness of the integrals was given in this way.

A coalgebra C is called left co-Frobenius if there exists a monomorphism of left C^*-modules from C to C^*. This is a generalization of the classical notion of Frobenius algebras. Also, C is called left semiperfect if the injective envelope $E(S)$ of any simple right C-comodule S is finite dimensional (see [19]). A left co-Frobenius coalgebra is left semiperfect, while the converse does not hold in general. However, for a Hopf algebra H, H is left (or right) co-Frobenius if and only if it is left (or right) semiperfect, and this is equivalent to H having non-zero integrals. This explains why Hopf algebras with non-zero integrals are called co-Frobenius Hopf algebras.

Information about a coalgebra C is captured by its coradical filtration $C_0 \subset C_1 \subset \dots$. Since the coradical filtration is a coalgebra filtration, one can construct the associated graded coalgebra gr C. Finiteness properties of the coalgebras C and gr C are similar. Our first general result is the following.

Theorem 0.1. *Let C be a coalgebra and gr C the graded coalgebra associated to the coradical filtration of C. Then C is left semiperfect if and only if so is gr C.*

Actually, we deduce Theorem 0.1 from a more specific result, relating the injective envelope over C of the simple right C-comodule S with the injective envelope over gr C of the simple right gr C-comodule S; see Theorem 1.3.

All the examples we know of co-Frobenius Hopf algebras have a *finite* coradical filtration. The main theme of this paper is: whether finiteness of the coradical filtration and existence of non-zero integrals, are equivalent conditions on a Hopf algebra. In one direction, a positive answer follows from the following

Theorem 0.2. *Let C be a coalgebra such that $C = C_n$ for some $n \geq 0$. Then the rational part of the right (or left) C^*-module C^* is non-zero.*

Corollary 0.3. *If H is a Hopf algebra such that there exists a non-negative integer n with $H_n = H$, then H is co-Frobenius.*

We conjecture that the opposite implication is also true: if H is co-Frobenius, then the coradical filtration is finite. It is known that the conjecture is true under the assumption that the coradical is a Hopf subalgebra, as shown by Radford [24]. We present an alternative proof in our main Theorem below. Corollary 0.3 allows to present new examples of co-Frobenius Hopf algebras. As a distinguished example, we prove that algebras of functions on quantum groups at roots of one are co-Frobenius. The proof uses the "Steinberg-type" decomposition theorem for quantum groups at roots of one, obtained by Lusztig [20]. This material is contained in Section 2.

An interesting particular situation is when the coradical H_0 of H is a Hopf subalgebra. Then the coradical filtration of H is a Hopf algebra filtration, gr H is a Hopf algebra and there exists a Hopf algebra projection $\pi :$ gr $H \to H_0$ which splits the inclusion of H_0 in gr H as the degree zero component. Then the subalgebra R of coinvariants of gr H with respect to the coaction of H_0 via π has a structure of a Hopf algebra in the category $_{H_0}^{H_0}\mathcal{YD}$ of Yetter-Drinfeld modules over H_0. Moreover R, which is called the *diagram* of H, is a graded subalgebra of gr H, and gr H can be reconstructed from R by bosonization, i.e. gr H is isomorphic to the biproduct $R\sharp H_0$. This lifting method to study Hopf algebras with the coradical a Hopf subalgebra was invented in [3], and consists in studying first all possible diagrams R, then transferring the information to gr H by bosonization, and finally by lifting the information from gr H to H. The method was essentially used only to study pointed Hopf algebras, where the coradical is a group algebra. In this paper we apply the method to any possible coradical. The main result of this paper is

Theorem 0.4. *Let H be a Hopf algebra with the coradical H_0 a Hopf subalgebra. Then the following assertions are equivalent.*

(1) *H is co-Frobenius.*
(2) *The associated graded Hopf algebra gr H is co-Frobenius.*
(3) *The diagram R of H is finite dimensional.*
(4) *The coradical filtration $H_0, H_1, \ldots$ of H is finite, i.e. there exists n such that $H_n = H$.*

This explains for instance why do we need to start with a necessarily finite dimensional quantum linear space in order to obtain by the lifting method a co-Frobenius Hopf algebra [7].

The implication $(1) \Rightarrow (4)$ in Theorem 0.4 was proved before in [24] with different methods. $(1) \Leftrightarrow (2)$ follows from Theorem 0.1, and $(4) \Rightarrow (1)$ is a particular case of Corollary 0.3.

The coradical of a Hopf algebra is a Hopf subalgebra if and only if the tensor product of two simple comodules is completely reducible. If this holds in a tensor category $\mathcal{C}$, then it is said that $\mathcal{C}$ has the Chevalley property. Furthermore, a Hopf algebra has the Chevalley property if the category of its finite dimensional modules has the Chevalley property. This denomination honors a classical result of Chevalley [12, p. 88]; it was proposed in [1], where triangular Hopf algebras with the Chevalley property were considered. See also [22].

The results we prove in this paper can be used on one hand to construct systematically co-Frobenius Hopf algebras from finite dimensional braided Hopf algebras over a cosemisimple Hopf algebra (the coradical), and eventually to classify classes of co-Frobenius Hopf algebras. We perform a preliminary discussion in Section 5.

But on the other hand, these results also serve as a test to decide that a certain Hopf algebra does not have the Chevalley property by looking to the dimension of the injective envelopes of simple comodules. See Section 4.

We work over a field k. The category of right (respectively left) comodules over a coalgebra C is denoted by $\mathcal{M}^C$ (respectively $^C\mathcal{M}$). If M is a right (or left) C-comodule, the injective

envelope of M in the category $\mathcal{M}^C$ (or $^C\mathcal{M}$), which exists since the category of comodules is a Grothendieck category, is denoted by $E(M)$. We refer to [26] and [14] for notation and facts about coalgebras, comodules and Hopf algebras.

1. The Loewy series of a comodule and the associated graded comodule

Let C be a coalgebra, $C_0 \subseteq C_1 \subseteq \ldots$ the coradical filtration of C, and $M \in \mathcal{M}^C$ a right C-comodule with comodule structure map $\rho : M \to M \otimes C$. The Loewy series $M_0, M_1, \ldots$ of M is defined as follows: $M_0 = s(M)$, the socle of M, i.e. M_0 is the sum of all simple subcomodules of M; hence for any $n \geq 0$, assume that we have defined M_n, then M_{n+1} is defined by $M_{n+1}/M_n = s(M/M_n)$. We obtain in this way a chain $M_0 \subseteq M_1 \subseteq \ldots$ of C-subcomodules of M. Since M is the sum of all its finite dimensional subcomodules we see that M is the union of all M_n's.

Lemma 1.1. *For any $n \geq 0$ we have that $M_n = \rho^{-1}(M \otimes C_n)$.*

Proof. Let J be the Jacobson radical of the dual algebra C^*, then $J = C_0^\perp$ and $M_n = ann_M(J^{n+1})$ for any $n \geq 0$ (see [14, Lemma 3.1.9]). If $m \in M$ such that $\rho(m) \in M \otimes C_n$, then since $C_n = (J^{n+1})^\perp$, we have that $c^* \cdot m = \sum c^*(m_1)m_0 = 0$ for any $c^* \in (J^{n+1})$, so then $m \in ann_M(J^{n+1}) = M_n$. Conversely, let $m \in M_n = ann_M(J^{n+1})$. Let $(m_i)_i$ be a basis of M and write $\rho(m) = \sum_i m_i \otimes c_i$ for some $c_i \in C$. Then for any $c^* \in J^{n+1}$ we have $0 = c^* \cdot m = \sum_i c^*(c_i)m_i$, so then $c^*(c_i) = 0$ for any i. Thus $c_i \in (J^{n+1})^\perp = C_n$ for any i, showing that $\rho(m) \in M \otimes C_n$. $\square$

The following result generalizes [26, Corollary 9.1.7], which states that the coradical filtration of a coalgebra is a coalgebra filtration.

Lemma 1.2. *For any $n \geq 0$ we have that $\rho(M_n) \subseteq \bigoplus_{i=0}^n M_i \otimes C_{n-i}$.*

Proof. We prove by induction on n. For $n = 0$, we have that M_0 is the socle of the C-comodule M, thus it is a sum of simple comodules. Since the subcoalgebra of C associated to any simple comodule is a simple subcoalgebra (see [14, Exercise 3.1.2]), we have that $\rho(M_0) \subseteq M_0 \otimes C_0$.

Assume now that the assertion holds for $j \leq n$. Let $(u_{0,i})_i$ be a basis for M_0. We complete this basis with a family $(u_{1,i})_i$ up to a basis of M_1, then proceed similarly by taking families $(u_{2,i})_i, \ldots, (u_{n+1,i})_i$ for obtaining basis of $M_2, \ldots, M_{n+1}$. Let $m \in M_{n+1}$. Then there exist uniquely determined families $(c_{0,i})_i, \ldots, (c_{n+1,i})_i$ of elements of C such that

$$\rho(m) = \sum_i u_{0,i} \otimes c_{0,i} + \ldots + \sum_i u_{n+1,i} \otimes c_{n+1,i}$$

Denote by $\overline{\rho} : M/M_n \to M/M_n \otimes C$ the comodule structure map of the factor comodule M/M_n, and by $\overline{x}$ the class modulo M_n of an element $x \in M$. Then $\overline{\rho}(\overline{m}) = \sum_i \overline{u_{n+1,i}} \otimes c_{n+1,i}$. On the other hand M_{n+1}/M_n is a sum of simple comodules, so $\overline{\rho}(\overline{m}) \in M_{n+1}/M_n \otimes C_0$, which shows that $c_{n+1,i} \in C_0$ for any i.

On the other hand, since $(\rho \otimes I)\rho = (I \otimes \Delta)\rho$, we have that

$$\sum_i \rho(u_{0,i}) \otimes c_{0,i} + \ldots + \sum_i \rho(u_{n+1,i}) \otimes c_{n+1,i} = \sum_i u_{0,i} \otimes \Delta(c_{0,i}) + \ldots + \sum_i u_{n+1,i} \otimes \Delta(c_{n+1,i})$$

By the induction hypothesis $\rho(u_{r,i}) \in \bigoplus_{p=0,r} M_p \otimes C_{r-p}$ for any $r \leq n$, so then by looking at the terms with the basis element $u_{r,i}$ on the first tensor position, where $r \leq n$, we see that

$$\Delta(c_{r,i}) \in C_0 \otimes C + C_1 \otimes C + \ldots + C_{n-r} \otimes C + C \otimes C_0 = C_{n-r} \otimes C + C \otimes C_0$$

This implies that $c_{r,i} \in C_{n-r} \wedge C_0 = C_{n-r+1}$, which ends the proof. $\square$

Now we construct the graded comodule associated to the Loewy series of a C-comodule M. In particular, for $M = C$, we obtain the graded coalgebra associated to the coradical filtration of C. For any $i \geq 1$ we denote by $\pi_i : C \to C/C_{i-1}$ and $p_i : M \to M/M_{i-1}$ the natural projections. We consider the spaces gr $C = \oplus_{i\geq 0} C_i/C_{i-1}$ and gr $M = \oplus_{i\geq 0} M_i/M_{i-1}$, where we take $C_{-1} = 0, M_{-1} = 0$. By Lemma 1.2, the map

$$(\oplus_{i=0}^n p_i \otimes \pi_{n-i}) \circ \rho : M \to \oplus_{i=0}^n (M/M_{i-1} \otimes C/C_{n-i-1})$$

induces a linear map

$$\rho_n : M_n \to \oplus_{i=0}^n (M_i/M_{i-1} \otimes C_{n-i}/C_{n-i-1})$$

Using again Lemma 1.2 we have that $\rho_n(M_{i-1}) = 0$, and thus ρ_n induces a linear map

$$\overline{\rho}_n : M_n/M_{n-1} \to \oplus_{i=0}^n (M_i/M_{i-1} \otimes C_{n-i}/C_{n-i-1})$$

If we regard $\oplus_{i=0}^n (M_i/M_{i-1} \otimes C_{n-i}/C_{n-i-1})$ as a subspace of gr $M \otimes$ gr C, the sum of all $\overline{\rho}_n$'s define a linear map $\overline{\rho} :$ gr $M \to$ gr $M \otimes$ gr C. This map can be described as follows. For $m \in M_n$, write $\rho(m) = \sum_{i=0}^n m_{0,i} \otimes m_{1,n-i}$, a Sigma Notation type representation, with the $m_{0,i}$'s lying in M_i and the $m_{1,n-i}$'s lying in C_{n-i}. Then we have that $\overline{\rho}(p_n(m)) = \sum p_i(m_{0,i}) \otimes \pi_{n-i}(m_{1,n-i})$; and this does not depend on the chosen representation of $\rho(m)$. In the case where $M = C$ and $\rho = \Delta$, we obtain the map $\overline{\Delta} :$ gr $C \to$ gr $C \otimes$ gr C, defined by $\overline{\Delta}(\pi_n(c)) = \sum \pi_i(c_{1,i}) \otimes \pi_{n-i}(c_{1,n-i})$ for $c \in C_n$ and any representation $\Delta(c) = \sum c_{1,i} \otimes c_{2,n-i}$ with the same convention as above. Straightforward computations show that gr C is a graded coalgebra with comultiplication $\overline{\Delta}$ and gr M is a graded right gr C-comodule via $\overline{\rho}$. We denote by gr $M(n) = M_n/M_{n-1}$ the homogeneous component of degree n of gr M.

The construction of the graded coalgebra associated to a coalgebra filtration of a coalgebra goes back to Sweedler's book (see [26, Section 11.1]). By [3, Lemma 2.3] we have that gr C is coradically graded, i.e. its coradical filtration is given by (gr $C)_n = \oplus_{i\leq n}$ gr $C(i)$ for any $n \geq 0$.

Theorem 1.3. *Let S be a simple right C-comodule, C any coalgebra. The graded comodule associated to the injective envelope over C of S, is isomorphic to the the injective envelope over gr C of the simple right gr C-comodule S.*

Proof. Since the socle of a direct sum of comodules is the sum of the socles of these co-modules, we see that for any family $(M(\lambda))_{\lambda \in \Lambda}$ of right C-comodules we have gr $(\oplus_{\lambda \in \Lambda} M(\lambda))$ $\simeq \oplus_{\lambda \in \Lambda}$ gr $M(\lambda)$ as right gr C-comodules. In particular if we write $C_0 = Soc(C) = \oplus_{S\in\mathcal{S}} S$ for some family $\mathcal{S}$ of simple right C-subcomodules of C, then $C = \oplus_{S\in\mathcal{S}} E(S)$ (see [14, Theorem

2.4.16]), so then gr $C = \oplus_{S \in \mathcal{S}}$ gr $E(S)$ as right gr C-comodules. In particular gr $E(S)$ is injective as a right gr C-comodule. Since $S = E(S)_0 = $ gr $E(S)(0) \subseteq $ gr $E(S)$, we obtain that $E_{\mathrm{gr}}(S) \subseteq $ gr $E(S)$, where $E_{\mathrm{gr}}(S)$ is an injective envelope of S as a right gr C-comodule.

On the other hand (gr $C)_0 = $ gr $C(0) = C_0 = \oplus_{S \in \mathcal{S}} S$ as right gr C-comodules, so then gr $C = \oplus_{S \in \mathcal{S}} E_{\mathrm{gr}}(S)$, showing that $E_{\mathrm{gr}}(S) = $ gr $E(S)$ for any $S \in \mathcal{S}$. $\square$

Proof of Theorem 0.1. Given a simple S, $E(S)$ is finite dimensional if and only if so is gr $E(S) \simeq E_{\mathrm{gr}}(S)$. $\square$

2. Coradical filtration of a co-Frobenius Hopf algebra

Let H be a Hopf algebra and $H_0, H_1, \ldots$ its coradical filtration. We first show that if the coradical filtration is finite, then necessarily H is co-Frobenius. This follows from Theorem 0.2, which gives information about coalgebras with finite coradical filtration.

Proof of Theorem 0.2. If $C = C_0$, then C is cosemisimple, therefore it is left and right co-Frobenius (see [14, Exercise 3.3.17]). In particular the rational part of C^* as a right (or left) C^*-module is non-zero. If C is not cosemisimple, let n be such that $C_{n-1} \neq C_n = C$. Then C/C_{n-1} is a semisimple right C-comodule, in particular it contains a maximal subcomodule. This induces a maximal right C-subcomodule X of C. The proof goes now as in [14, Proposition 3.2.2]. The natural projection $C \longrightarrow C/X$ produces an injective morphism of right C^*-modules $(C/X)^* \hookrightarrow C^*$. But C/X is simple, so it is finite dimensional and rational as a left C^*-module, and then $(C/X)^*$ is a rational right C^*-module (see [14, Lemma 2.2.12]). This shows that the rational part of the right C^*-module C^* is non-zero. $\square$

Corollary 0.3 follows from Theorem 0.2 and the fact that a Hopf algebra H is co-Frobenius if and only if $H^{*rat} \neq 0$ [14, Ch. 5]. Note that for a Hopf algebra the left and the right rational parts of H^* are equal, and they are denoted by H^{*rat}.

As said in the Introduction, we conjecture that the converse of Corollary 0.3 is true:

Conjecture 2.1. The coradical filtration of a co-Frobenius Hopf algebra is finite.

We would like to mention the following more precise question, which arose in joint work of Sonia Natale and the first-named author.

Question 2.2. Let H be a Hopf algebra and let T be a left integral on H. If H_m is different from H, is it true that T vanishes on H_m?

Note that a positive answer to Question 2.2 implies Conjecture 2.1. For $m = 0$, the answer is positive by Maschke's Theorem for Hopf algebras [27].

We now show how Corollary 0.3 allows to determine that some important Hopf algebras are co-Frobenius. We begin by the following general Lemma.

Lemma 2.3. *Let H be a Hopf algebra and let K be a Hopf subalgebra such that*

(2.1) $KH_0 \subseteq H_0$ *(in particular, $K \subseteq H_0$).*

(2.2) H *is of finite type as left module via multiplication over K.*

Then the coradical filtration of H is finite; and H is co-Frobenius.

Proof. By (2.1), each term H_n of the coradical filtration is a K-submodule; by (2.2), the coradical filtration is then finite. The last claim follows from Corollary 0.3. $\square$

In representation-theoretic terms, condition (2.1) means the following. If V is a simple K-comodule and W is a simple H-comodule then the H-comodule $V \otimes W$ is completely reducible. Indeed, let $\widehat{H}$ denote the set of (isomorphism classes of) simple comodules over a Hopf algebra H; we will confuss a class with a representant without danger. If U is any finite dimensional H-comodule, then let C_U denote the space of matrix coefficients of U; it is a subcoalgebra of H. We have $C_{U \otimes W} = C_U C_W$. Also $K = \sum_{V \in \widehat{K}} C_V$, $H_0 = \sum_{W \in \widehat{H}} C_W$, thus the claim.

We are led to the following definition.

Definition 2.4. Let H be a Hopf algebra with bijective antipode. Let $\widehat{H}_{\mathrm{SOC}}$ be the subset of $\widehat{H}$ consisting of all simple comodules V such that

(2.3) $\qquad\qquad\qquad V \otimes W$ and $W \otimes V$ are completely reducible

for all $W \in \widehat{H}$. The Hopf socle of H is

$$H_{\mathrm{SOC}} \ = \sum_{V \in \widehat{H}_{\mathrm{SOC}}} C_V.$$

Lemma 2.5. *Let H be a Hopf algebra with bijective antipode. The Hopf socle of H is a cosemisimple Hopf subalgebra of H.*

Proof. It is clear that H_{SOC} is a subcoalgebra of H_0. We prove that H_{SOC} is a subalgebra of H. Let $V, U \in \widehat{H}_{\mathrm{SOC}}$. Then $V \otimes U = \oplus_{1 \leq j \leq r} W_j$, where W_j are simple comodules. We have to show that $W_j \in \widehat{H}_{\mathrm{SOC}}$, $1 \leq j \leq r$. Let $W \in \widehat{H}$. Then $V \otimes U \otimes W$ is completely reducible; since $W_j \otimes W$ is a subcomodule of $V \otimes U \otimes W$, it is also completely reducible. By an analogous argument, $W \otimes W_j$ is completely reducible. Therefore, $W_j \in \widehat{H}_{\mathrm{SOC}}$. We finally prove that H_{SOC} is stable under the antipode $\mathcal{S}$. If V is a finite dimensional comodule, we denote by V^* be the comodule structure on the dual defined via $\mathcal{S}$, and by $^*V$, the comodule structure on the dual defined via $\mathcal{S}^{-1}$. We know that $(V \otimes W)^* \simeq W^* \otimes V^*$, $^*(V \otimes W) \simeq {}^*W \otimes {}^*V$ for any two finite dimensional comodules V and W. If $V \in \widehat{H}_{\mathrm{SOC}}$ and $W \in \widehat{H}$, then $^*(V^* \otimes W) \simeq {}^*W \otimes V$ is completely reducible, therefore $V^* \otimes W$ is completely reducible. Similarly, $W \otimes V^*$ is completely reducible. Thus, $V^* \in \widehat{H}_{\mathrm{SOC}}$, as needed. $\square$

Lemma 2.3 immediately implies:

Corollary 2.6. *Let H be a Hopf algebra with bijective antipode. If H is of finite type as left module via multiplication over H_{SOC}, then H is co-Frobenius.* $\square$

We ignore if the converse of Corollary 2.6 is true; that is, if any co-Frobenius Hopf algebra is of finite type over H_{soc} .

From now on, we assume that k is algebraically closed and has characteristic 0. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of rank n, and let G be the corresponding simply-connected algebraic group. Let $q \in k$ be a root of 1 of odd order $\ell \geq 3$, ℓ not divisible by 3 if $\mathfrak{g}$ contains a component of type G_2; and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra as defined in [20]. Recall the definition of modules of type 1 [20]. The following facts were proved by Lusztig:

- The set of isomorphism classes of simple $U_q(\mathfrak{g})$-modules of type 1 is parametrized by $(\mathbb{Z}_{\geq 0})^n$ [20, Prop. 6.4]. If $\lambda \in (\mathbb{Z}_{\geq 0})^n$, the corresponding simple module $L(\lambda)$ has highest weight λ.

- If $\lambda \in (\mathbb{Z}_{\geq 0})^n$, decompose it as $\lambda = \lambda' + \ell\lambda''$, where $\lambda', \lambda'' \in (\mathbb{Z}_{\geq 0})^n$ with each entry of λ' living in the interval $[0, \ell - 1]$. Then $L(\lambda) \simeq L(\lambda') \otimes L(\ell\lambda'')$ (Steinberg-type Theorem, [20, Th. 7.4]).

- There is an epimorphism of Hopf algebras $U_q(\mathfrak{g}) \to U(\mathfrak{g})$. If $\lambda'' \in (\mathbb{Z}_{\geq 0})^n$, the simple $U_q(\mathfrak{g})$-module $L(\ell\lambda'')$ inherits a structure of $U(\mathfrak{g})$; and as such, it is simple with highest weight λ'' [20, Prop. 7.5]. Denote this last module by $\underline{L}(\lambda'')$.

Let $k_q[G] \subset U_q(\mathfrak{g})^*$ be the Hopf algebra of matrix coefficients of modules of type 1. This is the algebra of functions on a quantum group at a root of one, as in [13]. The coradical of $k_q[G]$ is the span of the matrix coefficients of the simple modules $L(\lambda)$, $\lambda \in (\mathbb{Z}_{\geq 0})^n$. It is known that it is *not* a Hopf subalgebra.

The subalgebra of $k_q[G]$ spannned by the matrix coefficients of the simple modules $L(\ell\lambda'')$, $\lambda'' \in (\mathbb{Z}_{\geq 0})^n$, is a Hopf subalgebra of $k_q[G]$, isomorphic to the algebra $k[G]$ of regular functions on G. It is known that it is central [13].

Theorem 2.7. $k_q[G]$ *is co-Frobenius.*

Proof. Let $H = k_q[G]$, $K = k[G]$. (2.2) holds by [13]. Condition (2.1) is a consequence of the "Steinberg-type" decomposition theorem evoked above. Indeed, let $V = L(\ell\mu)$, $W = L(\lambda)$ with $\mu, \lambda \in (\mathbb{Z}_{\geq 0})^n$. Decompose λ as before, so that $L(\lambda) \simeq L(\lambda') \otimes L(\ell\lambda'')$, $\lambda', \lambda'' \in (\mathbb{Z}_{\geq 0})^n$. By Weyl's Theorem, $\underline{L}(\lambda'') \otimes \underline{L}(\mu)$ is completely reducible, say isomorphic to $\oplus_{j=1,\ldots,M}\underline{L}(\sigma_j)$. Then

$$W \otimes V \simeq L(\lambda') \otimes L(\ell\lambda'') \otimes L(\ell\mu) \simeq L(\lambda') \otimes (\oplus_{j=1,\ldots,M}L(\ell\sigma_j))$$

$$\simeq \oplus_{j=1,\ldots,M}L(\lambda') \otimes L(\ell\sigma_j) \simeq \oplus_{j=1,\ldots,M}L(\lambda' + \ell\sigma_j).$$

Therefore, $KH_0 = H_0K \subseteq H_0$. The Theorem follows now from Lemma 2.3. $\square$

Remark 2.8. It is known that $k_q[G]$ is an extension of $k[G]$ by a finite Hopf algebra, namely the dual of the corresponding Frobenius-Lusztig kernel. However, we do not known if it is a *cleft* extension; otherwise we could have deduced Theorem 2.7 from [8, 5.2]. It is however known that $k_q[G]$ is free over $k[G]$ [9, 10].

3. Co-Frobenius Hopf algebras whose coradical is a Hopf subalgebra

In this section we assume that the coradical H_0 of H is a Hopf subalgebra.

Write as in Section 1 $H_0 = \oplus_{S \in \mathcal{S}} S$ for some family $\mathcal{S}$ of simple left H-subcomodules of H. We identify gr H and $R\sharp H_0$, then gr $H = \oplus_{S \in \mathcal{S}} R\sharp S$.

Lemma 3.1. $R\sharp S$ is an injective left gr H-subcomodule of gr H for any $S \in \mathcal{S}$.

Proof. The comultiplication of the biproduct is $\Delta(r\sharp h) = \sum r^1\sharp(r^2)_{-1}h_1 \otimes (r^2)_0\sharp h_2$, where $r \mapsto \sum r^1 \otimes r^2$ is the comultiplication of the braided Hopf algebra R, and $r \mapsto \sum r_{-1} \otimes r_0$ is the left coaction of H_0 on R. This implies that $\Delta(R\sharp S) \subseteq$ gr $H \otimes (R\sharp S)$. The injectivity follows from the fact that $R\sharp S$ is a direct summand of the left gr H-comodule gr H. $\square$

Corollary 3.2. Let $S \in \mathcal{S}$ and let $E_{gr}(S)$ be the injective envelope of S as a left gr H-comodule. Then $E_{gr}(S) = R\sharp S$.

Proof. The result follows immediately from the previous lemma if we take into account that S is contained in $R\sharp S$, and that for any $S, T \in \mathcal{S}$, $S \neq T$, $(R\sharp S) \cap (R\sharp T) = 0$. $\square$

Proof of Theorem 0.4 (1) $\Leftrightarrow$ (2) follows from Theorem 0.1.

(2) $\Leftrightarrow$ (3) follows from Corollary 3.2.

(3) $\Rightarrow$ (4). Since R is finite dimensional there exists a positive

integer n such that $R = R(0) + \ldots + R(n)$. Then gr $H =$ gr $H(0) + \ldots +$ gr $H(n)$, implying that $H_n = H$.

(4) $\Rightarrow$ (1) follows from Corollary 0.3. $\square$

Here is an alternative proof of (4) $\Rightarrow$ (1). If the coradical filtration is finite, then the graded braided Hopf algebra R has a finite grading, *i.e.* $R = \oplus_{0 \leq n \leq N} R(n)$, with $R(N) \neq 0$. Since $R(0) = k$, any element $t \in R(N)$, $t \neq 0$ is an integral in R, see for instance [2, Prop. 3.2.2]. Hence R is finite dimensional by [15].

We state another consequence of Corollary 3.2. Let E be the injective envelope of the trivial H-comodule $k1$.

Corollary 3.3. Let $S \in \mathcal{S}$ and let $E(S)$ be the injective envelope of S as a left H-comodule. Then $E(S) \simeq E \otimes S$.

Proof. Observe first that the claim is true for gr H, by Corollary 3.2. Since $E \otimes S$ is a direct summand of $H \otimes S \simeq H^{\dim S}$, it is injective. Thus there exists a monomorphism of H-comodules $\varphi : E(S) \to E \otimes S$. As H_0 is a Hopf subalgebra of H, the Loewy series of $E \otimes S$ is $(E \otimes S)_n = E_n \otimes S$. Therefore, gr $\varphi :$ gr $E(S) \to$ gr $(E \otimes S) \simeq (\text{gr } E) \otimes S$ is an isomorphism. By a standard argument, φ is an isomorphism. $\square$

4. SOME APPLICATIONS

We give now some numerical criteria for the Chevalley property. All of them follow from Lemma 3.1.

Proposition 4.1. *Let H be a finite dimensional Hopf algebra with a simple right (or left) H-comodule S such that the dimension of S does not divide the dimension of $E(S)$. Then the category of finite dimensional H-comodules does not have the Chevalley property.* □

If the coradical of a Hopf algebra H is a Hopf subalgebra, then the diagram R is isomorphic to the injective envelope of the trivial gr H-comodule. Then:

Proposition 4.2. *Let H be a finite dimensional Hopf algebra such that the dimension of the injective envelope of the trivial H-comodule does not divide the dimension of H. Then the category of finite dimensional H-comodules does not have the Chevalley property.* □

Dualizing, we have:

Proposition 4.3. *Let H be a finite dimensional Hopf algebra with a simple right (or left) H-module S such that the dimension of S does not divide the dimension of the projective cover $P(S)$. Then H does not have the Chevalley property.* □

Proposition 4.4. *Let H be a finite dimensional Hopf algebra such that the dimension of the projective cover of the trivial H-module does not divide the dimension of H. Then H does not have the Chevalley property.* □

If C is a coalgebra and $M \in \mathcal{M}^C$ a right C-comodule, we can consider the Poincaré series of M, namely

$$\ell(M) = \sum_{n \geq 0} \ell_n(M) X^n \in \mathbb{Z}[[X]], \qquad \text{where} \quad \ell_n(M) = \dim M_n/M_{n-1}.$$

Clearly, $\ell(M) = \ell(\text{gr } M)$.

Proposition 4.5. *Let H be a co-Frobenius Hopf algebra with the coradical a Hopf subalgebra. If S is a simple right comodule, then the Poincaré polynomial of the injective envelope $E(S)$ satisfies a "Poincaré duality": $\ell_n(E(S)) = \ell_{top-n}(E(S))$, for all n, $0 \leq n \leq top$. Here "top" means the degree of the Poincaré polynomial.*

Proof. By Corollary 3.2, we have

$$\ell(E(S)) = \ell(\text{gr } E(S)) = \ell(R\#S) = \ell(R) \dim S.$$

Then the claim follows from the analogous claim for R, which is well-known [23]. □

5. Examples from the Lifting Method

We now explain how to obtain new examples, and eventually classification results, of co-Frobenius Hopf algebras via Theorem 0.4. We rely on the Lifting method [3, 4]; a detailed exposition is [6]. We fix a cosemisimple Hopf algebra K and seek for co-Frobenius Hopf algebras H with $H_0 \simeq K$.

Recall that a braided vector space is a pair (V, c), where V is a vector space (which we assume finite dimensional) and $c : V \otimes V \to V \otimes V$ is an isomorphism satisfying the braid equation. Given a braided vector space (V, c), there is a remarkable graded braided Hopf algebra $\mathfrak{B}(V)$ called the Nichols algebra of (V, c). The Lifting method in the present setting consists of the following steps.

(a). Determine when $\mathfrak{B}(V)$ is finite dimensional, for all braided vector spaces (V, c) arising as Yetter-Drinfeld modules over *some* cosemisimple Hopf algebra.

(b). For those V as in (a), find in how many ways, if any, they can be actually realized as Yetter-Drinfeld modules over our fixed K.

(c). For $\mathfrak{B}(V)$ as in (a), compute all Hopf algebras H such that gr $H \simeq \mathfrak{B}(V) \# K$ ("lifting").

(d). Investigate whether any *finite dimensional* graded braided Hopf algebra $R = \oplus_{n \geq 0} R(n)$ in $^K_K \mathcal{YD}$ satisfying $R(0) = k1$ and $P(R) = R(1)$, is generated by its primitive elements, *i. e.* is a Nichols algebra.

For the discussions of the different steps, we can profit previous investigations in the setting of finite dimensional Hopf algebras [3, 4].

Let us first discuss step (a). Actually, we have an ample supply of finite dimensional Nichols algebras $\mathfrak{B}(V)$. Let us recall some definitions from [3, 4].

- A braided vector space (V, c) is of *diagonal type* if V has a basis $x_1, \ldots, x_\theta$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$, for all $1 \leq i, j \leq \theta$, where q_{ij} are some scalars.

Since we are interested in finite dimensional Nichols algebras, we can (and shall) assume that $q_{ii} \neq 1$, for all i.

- A braided vector space of diagonal type is of *Cartan type* if $q_{ij} q_{ji} = q_{ii}^{a_{ij}}$, for all $1 \leq i \neq j \leq \theta$, where a_{ij} are some non-positive integers [4]. Set $a_{ii} = 2$ for all i.

The integers a_{ij} can be chosen so that the matrix (a_{ij}) is a generalized Cartan matrix. The following Theorem was proved in [4], from results of Lusztig and using the twisting operation.

- If (a_{ij}) is actually a finite Cartan matrix and the orders of the q_{ij}'s are odd and not divisible by 3, when the matrix (a_{ij}) has a component of type G_2, then $\mathfrak{B}(V)$ has finite dimension (which can be explicitly computed).

The simplest example is when $q_{ij}q_{ji} = 1$ for all $i \neq j$. Then $\mathfrak{B}(V)$ is called a finite quantum linear space; the examples in [7] are exactly liftings of quantum linear spaces over an arbitrary abelian group.

Let us comment on step (b) in the setting of the examples just discussed. Let us fix a braided vector space (V, c) of finite Cartan type, with the restrictions above. To realize (V, c) in $_K^K \mathcal{YD}$, for our fixed cosemisimple Hopf algebra K, we need:

- a family of characters $\chi_1, \ldots, \chi_\theta$ of K,
- and a family of central group-likes $g_1, \ldots, g_\theta \in G(K) \cap Z(K)$ such that

$$\chi_j(g_i) = q_{ij}, \quad 1 \leq i, j \leq \theta.$$

For instance, (V, c) can be realized as a Yetter-Drinfeld module over the group algebra of $\mathbb{Z}^s$, if $s \geq \theta$. To determine whether it can be realized as a Yetter-Drinfeld module over the group algebra of $\mathbb{Z}^s$, for $s < \theta$ is an interesting arithmetical problem.

Example 5.1. Let $s = 1$. If all the orders of the entries q_{ij} divide a fixed odd prime number $p > 3$, then necessarily $\theta \leq 2$ and the following restrictions on p are in force:

- If the Cartan matrix is of type A_2, then $p \equiv 1 \mod 3$.
- If the Cartan matrix is of type B_2, then $p \equiv 1 \mod 4$.
- If the Cartan matrix is of type G_2, then $p \equiv 1 \mod 3$.

Indeed, the image of $\mathbb{Z}$ in Aut V is a cyclic group of order p, and we can apply [4, Th. 1.3]. A similar discussion is also valid for $p = 3$, see *loc. cit.* We obtain in this way many new examples of co-Frobenius Hopf algebras, namely $\mathfrak{B}(V)\#k\mathbb{Z}$. The situation is somewhat different for $p = 2$ [23, 11]; the examples arising here are particular cases of those in [7].

The computation of the liftings of the Hopf algebras $\mathfrak{B}(V)\#K$, step (c) of the method, requires the knowledge of a presentation by generators and relations of $\mathfrak{B}(V)$. This was obtained in [5] for braidings of finite Cartan type. It is likely that the techniques of the finite dimensional case are also useful here, see [5, 6]. Finally, let us mention concerning step (d), that a positive result in this direction is given in [5].

Let us come back to the case $K = k\mathbb{Z}$. It is natural to ask whether there are Yetter-Drinfeld modules V over K which are *not* of finite Cartan type but such that $\mathfrak{B}(V)$ is finite dimensional. First, it can be shown that such V is necessarily of diagonal type. Second, a small quantity of examples of such V are known [23, 16].

Are there examples of Yetter-Drinfeld modules V over some cosemisimple Hopf algebra K which are *not* of diagonal type, but such that $\mathfrak{B}(V)$ is finite dimensional? Yes, a few ones; see [21, 16]. They can be realized over suitable group algebras.

On another direction, let K be the algebra of regular functions on a simple algebraic group. Then we do not know any example of Yetter-Drinfeld module V over K with $\mathfrak{B}(V)$ finite dimensional.

References

[1] N. Andruskiewitsch, P. Etingof and S. Gelaki, Triangular Hopf Algebras With The Chevalley Property, preprint (2000).

[2] N. Andruskiewitsch and M. Graña, Braided Hopf algebras over non-abelian groups, Bol. Acad. Ciencias (Córdoba) **63** (1999), 45-78.

[3] N. Andruskiewitsch, H. J. Schneider, Liftings of quantum linear spaces and pointed Hopf algebras of order p^3, J. Algebra **209** (1998), 658-691.

[4] _____, Finite quantum groups and Cartan matrices, Adv. Math. **154**, 1-45 (2000).

[5] _____, Finite quantum groups over abelian groups of prime exponent, preprint (1999).

[6] _____, Pointed Hopf algebras, to appear in "Recent developments in Hopf algebras", Cambridge U. Press.

[7] M. Beattie, S. Dăscălescu and L. Grünenfelder, Constructing pointed Hopf algebras by Ore extensions, J. Algebra **225** (2000), 743–770.

[8] M. Beattie, S. Dăscălescu, L. Grünenfelder and C. Năstăsescu, Finiteness conditions, co-Frobenius Hopf algebras, and quantum groups, J. Algebra **200** (1998), 312–333.

[9] K. A. Brown and I. Gordon, The ramifications of the centres: quantised function algebras at roots of unity, `math.RT/9912042`.

[10] K. A. Brown, I. Gordon and J.T. Stafford, $O_e(G)$ is a free module over $O(G)$, `math.QA/0007179`.

[11] S. Caenepeel and S. Dăscălescu, On pointed Hopf algebras of dimension 2^n, Bull. London Math. Soc. **31** (1999), pp. 17–24.

[12] C. Chevalley, Theory of Lie groups, v.III, 1951 (in French).

[13] C. De Concini, V. Lyubashenko, Quantum function algebra at roots of 1, Adv. Math. **108**, 205–262 (1994).

[14] S. Dăscălescu, C. Năstăsescu, Ş. Raianu, Hopf algebras: an introduction, Marcel Dekker.

[15] D. Fischman, S. Montgomery, H.-J. Schneider, Frobenius extensions of subalgebras of Hopf algebras, Trans. Amer. Math. Soc. **349** (1997), 4857-4895.

[16] M. Graña, On Nichols algebras of low dimension, Contemp. Math. **267** (2000), pp. 111–136.

[17] P. H. Hai,The integral on quantum supergroups of type $A_{r|s}$, `math.QA/9812036`.

[18] R. Larson and M. Sweedler, An associative orthogonal bilinear form for Hopf algebras, Amer. J. of Math. **91** (1969), pp. 75–94.

[19] B. I. P. Lin, Semiperfect coalgebras, J. Algebra **49** (1977), 357-373.

[20] G. Lusztig, Modular representations and quantum groups; in Classical groups and related topics (Beijing, 1987), Contemp. Math. **82** (1989), pp. 59–77.

[21] A. Milinski and H.-J. Schneider, Pointed Indecomposable Hopf Algebras over Coxeter Groups, in "New Trends in Hopf Algebra Theory"; Contemp. Math. **267** (2000), pp. 215–236.

[22] R. Molnar, Tensor products and semisimple modular representations of finite groups and restricted Lie algebras, Rocky Mountain J. Math. **11** (1981), 581–591.

[23] W.D. Nichols, Bialgebras of type one, Commun. Alg. **6** (1978), pp. 1521–1552.

[24] D. E. Radford, Finiteness conditions for a Hopf algebra with non-zero integrals, J. Alg. **46** (1977), 189-195.

[25] J. Sullivan, Affine group schemes with integrals, J. Algebra **22** (1972), 546-558.

[26] M. Sweedler, Hopf algebras, Benjamin, New York, 1969.

[27] _____, Integrals for Hopf algebras. Ann. of Math. (2) **89** (1969), 323-335.

[28] S. L. Woronowicz, Compact matrix pseudogroups, Comm. Math. Phys. **111** (1987), pp. 613–665.

N. A.: Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, (5000) Ciudad Universitaria, Córdoba, Argentina

E-mail address: `andrus@mate.uncor.edu`

S. D.: Facultatea de Matematica, University of Bucharest, Str. Academiei 14, RO-70109 Bucharest 1, Romania

E-mail address: `sdascal@al.math.unibuc.ro`